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The p-version of the finite element method for a singularly perturbed boundary value problem

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Abstract

The p-version of the finite element method is applied to solve the singularly perturbed two-point boundary value problem with or without turning point. With the special choice of mesh points, global error estimates are derived. In some cases, the exponential rate of convergence is obtained. Some numerical results are given to show the performance of the proposed method.

Keywords: Finite element method; p-version; Singularly perturbed

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1. Introduction

We consider the boundary value problem

$$\begin{aligned} &Lu \equiv -\varepsilon u'' - p(x)u' + q(x)u = f(x), \quad x \in (0,1), \\ &u(0) = u(1) = 0, \end{aligned} \tag{1.1}$$

where $\varepsilon \in (0,1)$ is a constant, $p(x), q(x), f(x)$ are sufficiently smooth functions on $[0,1]$. Moreover, they satisfy $q(x) \geqslant 0$ and $q(x) + \frac{1}{2}p'(x) \geqslant C > 0$. This kind of problems arises in many fields, for example, in convention diffusion equations for fluid mechanics in which the convective term dominates.

The variational problem corresponding to (1.1) is: Find $u \in H_0^1(0,1)$ such that $\forall v \in H_0^1(0,1)$

$$B(u,v) \equiv \int_0^1 (\varepsilon u'v' - pu'v + quv)\,\mathrm{d}x = \int_0^1 fv\,\mathrm{d}x \tag{1.2}$$

holds. Throughout the paper $H^s(I)$ denotes the usual Sobolev space on interval I and $H_0^1(0,1) = \{v \mid v \in H^1(0,1), v(0) = v(1) = 0\}$, C is a constant dependent on $p(x), q(x), f(x)$, but independent of

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ε, not necessarily the same in each occurrence, sometimes with an integer as its subscript just to emphasize the dependence relationship.

For problem (1.1) we consider two cases varying with $p(x)$. One is the nonturning point case in which $p(x) \equiv 0$ or $p(x) \neq 0, \forall x \in [0,1]$. When $p(x) > 0$ on $[0,1]$ there is a boundary layer at $x=0$; $p(x) \equiv 0$ and there are boundary layers at $x=0$ and $x=1$. The other is the turning point case in which $p(x)$ has at least a zero point in $(0,1)$, here for simplicity $p(x)$ is allowed to have one simple zero located at $x=0.5$. Now besides the boundary layer at $x=0$ or 1, there is the so-called internal layer at $x=0.5$. For all cases, it is well known that obtaining an accurate numerical solution of problem (1.1) is sometimes difficult when parameter ε is very small. Much attention has been focused on the construction of computational methods for such kinds of problems. Ordinarily, the methods are required to have uniform accuracy with respect to ε, or in other words, do not deteriorate as $\varepsilon \to 0$. Generally, they are in two classes, finite difference method and finite element method, see [3–9, 12–17, 19, 20] and references therein.

It is known to us that conventional h-version finite element method does not work well for singularly perturbed problems, particularly in the boundary layer region, unless the mesh spacing $h < \varepsilon$, but practically it is impossible. Schatz and Wahlbin give elaborate analysis on its performance when applying to second-order singular perturbation problems in [16, 20]. A fourth-order problem is studied in [17]. In order to obtain better performance, many authors use Petrolev–Galerkin methods, that is, the trial space and the test space can be different. In [3, 19], Babuška and Szymczak propose a method and prove its quasi-optimality. Based on this, they derive an adaptive method. On the other hand, uniform convergence can be obtained by adding singular functions of boundary layer type to the test or trial spaces (see [7, 12, 14, 15]). The construction of these spaces is deeply related to the original differential equations and is difficult to generalize to higher-dimensional problems.

Now we give a new approach to handle problem (1.1). Let $\Delta \equiv \{0=x_0 < x_1 < \cdots < x_n < x_{n+1}=1\}$, where n will be shown to be in direct proportion to $\log 1/\varepsilon$ and the distribution of mesh points vary with the cases given above. More precisely, proper mesh refinement is done in boundary layer or internal layer regions. It is very similar to those in [10, 11]. Actually, boundary or internal layer behavior is very similar to singularity behavior of differential equations. Denote $I_i = [x_{i-1}, x_i]$ $(i = 1, \ldots, n+1)$ and define

$$\begin{aligned} &P_p(I_i) = \{v \mid v \quad \text{is a polynomial of degree } p \text{ on } I_i\},\\ &S(p,\sigma,n) = \{v \mid v|_{I_i} \in P_p(I_i),\ v \text{ is continuous on } [0,1]\}, \\ &S_p = \{v \mid v \in S(p,\sigma,n),\ v(0) = v(1) = 0\}. \end{aligned} \tag{1.3}$$

So S_p is a finite-dimensional subspace of $H_0^1(0,1)$. Then the finite element solution $u_p \in S_p$ for (1.2) is defined by

$$B(u_p, v) = \int_0^1 f v \,\mathrm{d}x, \quad \forall v \in S_p. \tag{1.4}$$

Once Δ is fixed, the convergence will be achieved by increasing p. For more about the p-version of the finite element method, see [2] and references therein.

This is a kind of conventional finite element method. There are several h–p version programs on the market, so we can employ them to compute numerical solutions. Based on the analysis in [18], it is our opinion that this method can be applied to higher-dimensional problems. Actually, in [1]

Babuška and Li obtain many numerical results by this kind of method for plate bending problem which is singularly perturbed. We will show that it has a high rate of convergence. Also the L^∞-convergence is obtained inside the boundary layer. Although it is not uniformly convergent since n in Δ depends on ε, it is practically useful.

Introduce several norms

$$\|u\|_1 = \left\{\int_0^1 (u'^2 + u^2)\,\mathrm{d}x\right\}^{1/2}, \quad \|u\|_0 = \left\{\int_0^1 u^2\,\mathrm{d}x\right\}^{1/2},$$

$$\|u\|_{1,\varepsilon} = \left\{\int_0^1 (\varepsilon u'^2 + u^2)\,\mathrm{d}x\right\}^{1/2}, \quad \|u\|_{1,\varepsilon,1/\varepsilon} = \left\{\int_0^1 \left(\varepsilon u'^2 + \frac{1}{\varepsilon}u^2\right)\,\mathrm{d}x\right\}^{1/2}.$$

From (1.2) and (1.4) we have

$$B(u-u_p, u-u_p) = B(u-u_p, u-v), \quad \forall v \in S_p. \tag{1.5}$$

The outline of this paper is as the follows. In Section 2 we consider the nonturning point case, the emphasis is laid on the case when $p(x)$ is strictly positive on $[0,1]$. In Section 2.1 we obtain a decomposition of the exact solution through asymptotic analysis technique, global error estimates are presented in Section 2.2. In Section 3 we consider the turning point case. To show the performance of our proposed method, several numerical experiments are given in Section 4, where we may find that sometimes even the exponential rate of convergence is obtained.

2. The nonturning point case

First $p(x)$ is supposed to be positive on $[0,1]$, then for bilinear form $B(u,v)$ on $H^1(0,1)\times H^1(0,1)$, it is obvious that

$$\begin{aligned} &B(u,u) \geqslant C\|u\|_{1,\varepsilon}^2,\\ &|B(u,v)| \leqslant C\|u\|_{1,\varepsilon}\|v\|_{1,\varepsilon,1/\varepsilon},\\ &|B(u,v)| \leqslant C\|u\|_{1,\varepsilon}\|v\|_1, \end{aligned} \tag{2.1}$$

in the last inequality of (2.1), either u or v must be in $H_0^1(0,1)$. Now in Δ let $x_i = \sigma^{n+1-i}$ $(i=1,\dots,n)$ where $\sigma\in(0,1)$ is a constant and n is dependent on ε by the relation which will be given later. Such kind of discretization was originally proposed in [10] to handle the singularity problem.

2.1. Decomposition of the exact solution

In this part we want to get the decomposition of u, the solution of (1.1), which forms the base of our error estimation.

Our immediate goal is to develop approximation of u by sums of the form

$$u \sim u^I + u^B := \sum_{i=0}^{+\infty} u_{I,i}(x)\varepsilon^i + \sum_{i=0}^{+\infty} u_{B,i}(\hat{x})\varepsilon^i,$$

where $\hat{x} = x/\varepsilon$. First we give a formal calculation to motivate appropriate definitions of $u_{I,i}(x)$ and $u_{B,i}(\hat{x})$. Later we will give rigorous bound for the error in the asymptotic expansions.

Inserting the series expansion for u^I in $Lu^I = f, u^I(1) = 0$ and equating coefficients of corresponding powers of ε, we obtain the differential equations defining $u_{I,i}$,

$$\begin{aligned} &-p(x)u'_{I,0}(x) + q(x)u_{I,0}(x) = f(x), \\ &-p(x)u'_{I,i}(x) + q(x)u_{I,i}(x) = u''_{I,i-1}(x), \quad i = 1, 2, \ldots \end{aligned} \tag{2.2}$$

and the boundary conditions

$$u_{I,i}(1) = 0, \quad i = 0, 1, \ldots \tag{2.3}$$

Usually, $u_{I,i}(x)$ cannot satisfy the boundary condition $u_{I,i}(0) = 0$. In order to obtain the defining problems for the boundary correctors $u_{B,i}(\hat{x})$, we use the variable transformation from x to $\hat{x}$, then

$$Lu^B = -\frac{1}{\varepsilon}\frac{\mathrm{d}^2u^B}{\mathrm{d}\hat{x}^2} - \frac{p(\hat{x}\varepsilon)}{\varepsilon}\frac{\mathrm{d}u^B}{\mathrm{d}\hat{x}} + q(\hat{x}\varepsilon)u^B. \tag{2.4}$$

For $p(x)$ and $q(x)$ we get the formal Taylor series expansions,

$$p(\hat{x}\varepsilon) = \sum_{i=0}^{+\infty}(\hat{x}\varepsilon)^i \frac{p^{(i)}(0)}{i!}, \qquad q(\hat{x}\varepsilon) = \sum_{i=0}^{+\infty}(\hat{x}\varepsilon)^i \frac{q^{(i)}(0)}{i!}. \tag{2.5}$$

We now calculate the differential equations determining $u_{B,i}(\hat{x})$ by inserting the series expansion for u^B in (2.4), using (2.5) and equating coefficients of corresponding powers of ε. So we obtain from (2.4) the equations

$$\begin{aligned} &u''_{B,0}(\hat{x}) + p(0)u'_{B,0}(\hat{x}) = 0, \\ &u''_{B,i}(\hat{x}) + \sum_{k=0}^{i} u'_{B,k}(\hat{x})\frac{\hat{x}^{i-k}p^{(i-k)}(0)}{(i-k)!} = \sum_{k=0}^{i-1} u_{B,k}(\hat{x})\frac{\hat{x}^{i-1-k}q^{(i-1-k)}(0)}{(i-1-k)!}, \quad i = 1, 2, \ldots \end{aligned} \tag{2.6}$$

Inserting the expansions for u^I and u^B in $u^I(0) + u^B(0) = 0$ and matching powers, we obtain the boundary conditions

$$u_{B,i}(0) = -u_{I,i}(0). \tag{2.7}$$

Finally, in order to determine $u_{B,i}(\hat{x})$ uniquely, we also impose the conditions at infinity

$$\lim_{\hat{x}\to+\infty} u_{B,i}(\hat{x}) = 0. \tag{2.8}$$

So $u_{B,i}(\hat{x})$ is uniquely defined by (2.6)–(2.8), from which we get

$$u_{B,i}(\hat{x}) = r_i(\hat{x})\mathrm{e}^{-p(0)\hat{x}}, \tag{2.9}$$

where $r_i(\hat{x})$ is a polynomial.

Let

$$R(x) = u(x) - \sum_{i=0}^{N} u_{I,i}(x)\varepsilon^i - \sum_{i=0}^{N} u_{B,i}(\hat{x})\varepsilon^i,$$

where N is an arbitrary given positive integer, $u_{I,i}(x)$ is defined by (2.2), (2.3) and $u_{B,i}(\hat{x})$ by (2.6)–(2.8). For $p(x)$ and $q(x)$, we have

$$\begin{aligned} p(\hat{x}\varepsilon) &= \sum_{i=0}^{N}(\hat{x}\varepsilon)^i \frac{p^{(i)}(0)}{i!} + \int_0^{\hat{x}\varepsilon} \frac{(\hat{x}\varepsilon-\xi)^N}{N!}\frac{\mathrm{d}^{(N+1)}p(\xi)}{\mathrm{d}\xi^{N+1}}\,\mathrm{d}\xi, \\ q(\hat{x}\varepsilon) &= \sum_{i=0}^{N}(\hat{x}\varepsilon)^i \frac{q^{(i)}(0)}{i!} + \int_0^{\hat{x}\varepsilon} \frac{(\hat{x}\varepsilon-\xi)^N}{N!}\frac{\mathrm{d}^{(N+1)}q(\xi)}{\mathrm{d}\xi^{N+1}}\,\mathrm{d}\xi. \end{aligned} \tag{2.10}$$

Using (2.9), (2.10) and noting that $\mathrm{e}^{-(p(0)/2)\hat{x}}\hat{x}^i \leqslant C_i$, through tedious calculation, we get

$$LR(x) = \varepsilon^{N+1}g(x),$$

where $g(x)$ satisfies

$$\left|\frac{\mathrm{d}^i g(x)}{\mathrm{d}x^i}\right| \leqslant C_N(1+\varepsilon^{-i-1}\mathrm{e}^{-(p(0)/2)x/\varepsilon}), \quad i=0,1,\ldots,N.$$

It is obvious that $R(0)=0$ and $|R(1)| \leqslant C_N\varepsilon^{N+1}$. Using the results in [13], for $R(x)$ we have

$$\left|\frac{\mathrm{d}^i R(x)}{\mathrm{d}x^i}\right| \leqslant C_N(1+\varepsilon^{-i}\mathrm{e}^{-(p(0)/2)x/\varepsilon})\varepsilon^{N+1}, \quad i=0,1,\ldots,N+1. \tag{2.11}$$

Till now we got the following decomposition for u:

$$u = u_N^I + u_N^B, \tag{2.12}$$

where $u_N^B = \sum_{i=0}^N u_{B,i}(\hat{x})\varepsilon^i$, $u_N^I = \sum_{i=0}^N u_{I,i}(x)\varepsilon^i + R(x)$. From (2.11) we have

$$\left|\frac{\mathrm{d}^i u_N^I}{\mathrm{d}x^i}\right| \leqslant C_N, \quad i=0,1,\ldots,N+1. \tag{2.13}$$

2.2. Error estimates

The idea of our error estimates is due to Guo and Babuška [11]. There is something similar between singularity behavior and boundary layer behavior.

Lemma 1. *If $u\in H^{k+1}(I)$ where k is a positive integer and $I=[-1,1]$, then there exists a polynomial $\phi(x)$ of degree k, such that for $m=0,1$,*

$$\left\|\frac{\mathrm{d}^m(u-\phi)}{\mathrm{d}x^m}\right\|_{L^2(I)}^2 \leqslant C\frac{(k-s)!}{(k+s)!}\left\|\frac{\mathrm{d}^{s+1}u}{\mathrm{d}x^{s+1}}\right\|_{L^2(I)}^2, \tag{2.14}$$

where $0\leqslant s\leqslant k$; moreover $\phi(-1)=u(-1), \phi(1)=u(1)$.

Proof. Let $L_i(x)$ be a Legendre polynomial on I; then we have

$$\int_{-1}^{1}\frac{\mathrm{d}^\alpha L_i}{\mathrm{d}x^\alpha}\frac{\mathrm{d}^\alpha L_j}{\mathrm{d}x^\alpha}(1-x^2)^\alpha\,\mathrm{d}x = \begin{cases} \dfrac{2}{2i+1}\dfrac{(i+\alpha)!}{(i-\alpha)!}, & \alpha\leqslant i \text{ and } i=j, \\ 0, & \text{otherwise.}\end{cases} \tag{2.15}$$

Since $u \in H^{k+1}(I)$, we have the following expansion:

$$\frac{du}{dx} = \sum_{i=0}^{+\infty} c_i L_i(x), \quad c_i = \frac{2i+1}{2} \int_{-1}^{1} \frac{du}{dx} L_i(x)\,dx. \tag{2.16}$$

Let $\phi(x) = \int_{-1}^{x} \sum_{i=0}^{k-1} c_i L_i(\xi) d\xi + u(-1)$; then $\phi(x)$ is a polynomial of degree k and it is obvious that $\phi(-1) = u(-1), \phi(1) = u(1)$. Moreover, we have $du/dx - (d\phi/dx) = \sum_{i=k}^{+\infty} c_i L_i(x)$, then we get

$$\left\| \frac{du}{dx} - \frac{d\phi}{dx} \right\|_{L^2(I)}^2 \leqslant \sum_{i=k}^{+\infty} c_i^2 \frac{2}{2i+1} \leqslant \frac{(k-s)!}{(k+s)!} \sum_{i=s}^{+\infty} c_i^2 \frac{2}{2i+1} \frac{(i+s)!}{(i-s)!}. \tag{2.17}$$

From (2.15) and (2.16) we get

$$\int_{-1}^{1} \left(\frac{d^{s+1}u}{dx^{s+1}} \right)^2 (1-x^2)^s\,dx = \sum_{i=s}^{+\infty} c_i^2 \frac{2}{2i+1} \frac{(i+s)!}{(i-s)!},$$

so (2.14) holds from (2.17) when $m = 1$. From $\int_{-1}^{x} L_n(\xi)\,d\xi = (1/(2n+1))(L_{n+1}(x) - L_{n-1}(x))$, we know that (2.14) holds when $m = 0$. □

By the scaling argument we obtain the following lemma.

Lemma 2. *If $u \in H^{k+1}(a,b)$, then there exists a polynomial $\phi(x)$ of degree k, such that for $m = 0, 1$,*

$$\left\| \frac{d^m(u-\phi)}{dx^m} \right\|_{L^2(a,b)}^2 \leqslant C \frac{(k-s)!}{(k+s)!} \left(\frac{b-a}{2} \right)^{2(s+1-m)} \left\| \frac{d^{s+1}u}{dx^{s+1}} \right\|_{L^2(I)}^2,$$

where $0 \leqslant s \leqslant k$ and $\phi(a) = u(a), \phi(b) = u(b)$.

Denote

$$S = \{u | u \in C^\infty(0,1), \left| \frac{d^k u}{dx^k} \right| \leqslant C \left(\frac{d}{\varepsilon} \right)^k e^{-(\alpha x/\varepsilon)}, \quad k = 0, 1, \ldots\}, \tag{2.18}$$

where d and α are constants. Then if $u \in S$, the following estimates hold:

$$\int_0^1 \left| \frac{d^k u}{dx^k} \right|^2 x^{2(k-l)}\,dx \leqslant C \left(\frac{d}{\alpha} \right)^{2k} \varepsilon^{1-2l} [(k-l)!]^2, \quad l = 0, 1. \tag{2.19}$$

Lemma 3. *If $u \in S$, then there exists $\phi(x) \in S(p, \sigma, n)$, such that $\phi(0) = u(0), \phi(1) = u(1)$ and*

$$\left\| \frac{d^m(u-\phi)}{dx^m} \right\|_0 \leqslant C\delta^p \varepsilon^{1/2-m}, \quad m = 0, 1, \tag{2.20}$$

$$\left\| x \frac{d(u-\phi)}{dx} \right\|_0 \leqslant C\delta^p \varepsilon^{1/2}, \tag{2.21}$$

where $\delta \in (0,1)$ is a constant. A condition on n has to be imposed and will be given in the proof.

Proof. Applying Lemma 2, we know that there exists $\phi_1(x) \in P_p(I_1)$, such that for $m = 0, 1$,

$$\left\| \frac{\mathrm{d}^m(u-\phi_1)}{\mathrm{d}x^m} \right\|^2_{L^2(I_1)} \leqslant C \frac{1}{(2p)!} \left(\frac{\sigma^n}{2}\right)^{2(p+1-m)} \left\| \frac{\mathrm{d}^{p+1}u}{\mathrm{d}x^{p+1}} \right\|^2_{L^2(I_1)}.$$

Noting (2.18), we now get

$$\left\| \frac{\mathrm{d}^m(u-\phi_1)}{\mathrm{d}x^m} \right\|^2_{L^2(I_1)} \leqslant C \frac{1}{(2p)!} \left(\frac{\sigma^n d}{2\varepsilon}\right)^{2(p+1-m)} \varepsilon^{1-2m}. \tag{2.22}$$

Moreover, $\phi_1(0) = u(0), \phi_1(x_1) = u(x_1)$.

Applying Lemma 2 repeatedly on $I_i (i=2,\dots,n+1)$, we know that for u, there exists $\phi_i(x) \in P_p(I_i)$ such that $\phi_i(x_{i-1}) = u(x_{i-1})$, $\phi_i(x_i) = u(x_i)$ and further

$$\left\| \frac{\mathrm{d}^m(u-\phi_i)}{\mathrm{d}x^m} \right\|^2_{L^2(I_i)} \leqslant C \frac{(p-s)!}{(p+s)!} \left(\frac{x_i - x_{i-1}}{2}\right)^{2(s+1-m)} \left\| \frac{\mathrm{d}^{s+1}u}{\mathrm{d}x^{s+1}} \right\|^2_{L^2(I_i)}, \tag{2.23}$$

where $s = 0, \dots, p$.

Now we give some modification on the right-hand side of (2.23),

$$\begin{aligned} \left\| \frac{\mathrm{d}^{s+1}u}{\mathrm{d}x^{s+1}} \right\|^2_{L^2(I_i)} &= \int_{x_{i-1}}^{x_i} \left| \frac{\mathrm{d}^{s+1}u}{\mathrm{d}x^{s+1}} \right|^2 x^{2(s+1-m)} x^{-2(s+1-m)} \,\mathrm{d}x \\ &\leqslant (x_{i-1})^{-2(s+1-m)} \int_{x_{i-1}}^{x_i} \left| \frac{\mathrm{d}^{s+1}u}{\mathrm{d}x^{s+1}} \right|^2 x^{2(s+1-m)} \,\mathrm{d}x. \end{aligned}$$

Define $\lambda = (1-\sigma)/\sigma$, and by applying (2.19) and (2.23), we get

$$\sum_{i=2}^{n+1} \left\| \frac{\mathrm{d}^m(u-\phi_i)}{\mathrm{d}x^m} \right\|^2_{L^2(I_i)} \leqslant C \frac{(p-s)!}{(p+s)!} \left(\frac{\lambda d}{2\alpha}\right)^{2(s+1-m)} [(s+1-m)!]^2 \varepsilon^{1-2m}. \tag{2.24}$$

For Gamma function $\Gamma(x)$, we get that for any integer $k \geqslant 0$ and $\theta \in [0,1)$, there exists constants C_1 and C_2 such that $C_1 \Gamma(k+1+\theta) \leqslant k!(k+1)^\theta \leqslant C_2 \Gamma(k+1+\theta)$. Noting $k!(k+1)^\theta = (k!)^{1-\theta}((k+1)!)^\theta$, so in (2.24) we can extend s to be any real number in $[0,p]$.

Now we construct $\phi(x) \in S(p,\sigma,n)$ by letting $\phi(x) = \phi_i(x)$ on $I_i (i=1,\dots,n+1)$; then (2.22) and (2.24) give

$$\begin{aligned} \left\| \frac{\mathrm{d}^m(u-\phi)}{\mathrm{d}x^m} \right\|^2_0 \leqslant C\varepsilon^{1-2m} \Bigg\{ &\left(\frac{\sigma^n d}{2\varepsilon}\right)^{2(p+1-m)} \frac{1}{(2p)!} \\ &+ \frac{\Gamma(p-s+1)}{\Gamma(p+s+1)} \left(\frac{\lambda d}{2\alpha}\right)^{2(s+1-m)} [\Gamma(s+2-m)]^2 \Bigg\}. \end{aligned} \tag{2.25}$$

In the above inequality, let $s = \beta p$ ($\beta \in [0,1]$); by Stirling's formula, we get

$$\left\| \frac{\mathrm{d}^m(u-\phi)}{\mathrm{d}x^m} \right\|_0 \leqslant C\varepsilon^{1/2-m} \left\{ \left(\frac{\sigma^n d}{2\varepsilon}\right)^p \left(\frac{1}{(2p)!}\right)^{1/2} + (F(\beta))^{p/2} \right\} \max\left(\frac{\sigma^n d}{2\varepsilon}, \frac{\lambda d}{2\alpha}, 1\right), \tag{2.26}$$

where

$$F(\beta)=\frac{(1-\beta)^{1-\beta}}{(1+\beta)^{1+\beta}}\left(\frac{\lambda d\beta}{2\alpha}\right)^{2\beta}.$$

Define $F_{\min}=\min_{\beta\in[0,1]}F(\beta)$, and by a long calculation, we get

$$F_{\min}=F(\beta_{\min})=\left(\frac{\lambda d}{\sqrt{(\lambda d)^2+4\alpha^2}+2\alpha}\right)^2, \tag{2.27}$$

where $\beta_{\min}=2\alpha/(\sqrt{4\alpha^2+(\lambda d)^2})$. We give a condition on n which has to be

$$n\geqslant\left[\frac{\log\varepsilon+\frac{1}{2}\log F_{\min}-\log\frac{d}{2}}{\log\sigma}\right]+1, \tag{2.28}$$

where $[x]$ denotes the maximal integer less than x. Under (2.28) we have $\sigma^n d/2\varepsilon\leqslant\sqrt{F_{\min}}$, so in (2.26) letting $\beta=\beta_{\min}$, we get

$$\left\|\frac{\mathrm{d}^m(u-\phi)}{\mathrm{d}x^m}\right\|_0\leqslant C\varepsilon^{1/2-m}\left(\sqrt{F_{\min}}\right)^p.$$

(2.21) can be obtained by the above process with a slight modification. From (2.22) we have

$$\left\|x\frac{\mathrm{d}(u-\phi_1)}{\mathrm{d}x}\right\|^2_{L^2(I_1)}\leqslant C\frac{\varepsilon}{(2p)!}\left(\frac{\sigma^n d}{2\varepsilon}\right)^{2(p+1)}.$$

From (2.23) we get

$$\begin{aligned}\left\|x\frac{\mathrm{d}(u-\phi_i)}{\mathrm{d}x}\right\|^2_{L^2(I_i)}&\leqslant C\,(x_i)^2\frac{(p-s)!}{(p+s)!}\left(\frac{x_i-x_{i-1}}{2}\right)^{2s}\left\|\frac{\mathrm{d}^{s+1}u}{\mathrm{d}x^{s+1}}\right\|^2_{L^2(I_i)}\\&\leqslant\frac{C}{\sigma^2}\frac{(p-s)!}{(p+s)!}\left(\frac{\lambda d}{2\alpha}\right)^{2s}\int_{x_{i-1}}^{x_i}\left|\frac{\mathrm{d}^{s+1}u}{\mathrm{d}x^{s+1}}\right|^2x^{2s+2}\,\mathrm{d}x.\end{aligned}$$

What follows is the same as above.

We finish our proof by letting $\delta=\sqrt{F_{\min}}$. □

Lemma 3 implies that by proper choice of mesh points, piecewise polynomial can approximate boundary layer type function well. Now we give the estimation on $u-u_p$. Noting that (2.9) holds, we know that u_N^B in (2.12) belongs to S defined by (2.18). By Lemma 3, for u_N^B, there exists $\phi^B(x)\in S(p,\sigma,n)$ such that $\phi^B(0)=u_N^B(0),\phi^B(1)=u_N^B(1)$ and

$$\left\|\frac{\mathrm{d}^m(u_N^B-\phi^B)}{\mathrm{d}x^m}\right\|_0\leqslant C\varepsilon^{1/2-m}\delta^p. \tag{2.29}$$

For u_N^I in (2.12), using the method in [2] and (2.13), we obtain $\phi^I(x)\in S(p,\sigma,n)$ such that $\phi^I(0)=u_N^I(0),\phi^I(1)=u_N^I(1)$ and

$$\|u_N^I-\phi^I\|_1\leqslant C_N p^{-N}. \tag{2.30}$$

Because $\phi^I + \phi^B \in S_p$, using (1.5) and (2.1), we have $\|u - u_p\|_{1,\varepsilon} \leqslant C(\|u_N^I - \phi^I\|_1 + \|u_N^B - \phi^B\|_{1,\varepsilon,1/\varepsilon})$. Combining it with (2.29) and (2.30), we get the following theorem.

Theorem 4. *Let u and u_p be the solutions of* (1.2) *and* (1.4), *respectively. Suppose* (2.28) *holds, then we have the error estimation*

$$\|u - u_p\|_{1,\varepsilon} \leqslant C_N(\delta^p + p^{-N}). \tag{2.31}$$

Remark. We will discuss the choice of σ in Section 4. Once σ is given, (2.28) implies that the smaller ε is, the larger n is, but n grows slowly and is only directly proportional to $\log 1/\varepsilon$. If the exact solution satisfies (2.19), the second term on the right-hand side of (2.31) vanishes and the exponential rate of convergence can be obtained.

With regard to the L^∞-estimation for $u - u_p$, it seems difficult. We can get nothing but the following. From $u(x) - u_p(x) = \int_0^x (u'(t) - u_p'(t))\,\mathrm{d}t$ and the Hölder inequality we obtain

$$|u(x) - u_p(x)| \leqslant \sqrt{x}\|(u - u_p)'\|_0 \leqslant \sqrt{\frac{x}{\varepsilon}}\|u - u_p\|_{1,\varepsilon}, \tag{2.32}$$

which means that in the tiny neighborhood of $x = 0$, or inside the boundary layer, pointwise error bound which is independent of ε is obtained.

On the whole domain, by the use of the one-dimensional Sobolev inequality and (2.31), we get

$$|u - u_p|_\infty \leqslant C\|u - u_p\|_1^{1/2}\|u - u_p\|_0^{1/2} \leqslant \frac{C}{\varepsilon^{1/4}}\|u - u_p\|_{1,\varepsilon},$$

which is useless for the very small ε. But in our practical computation, pointwise convergence can be seen even outside the boundary layer.

When $p(x) \equiv 0$, we can use the same method as above. Now there are boundary layers at $x = 0$ and $x = 1$, so the mesh refinements must be done near both $x = 0$ and $x = 1$. More precisely, in Δ, let $n = 2k + 1$, $x_i = \sigma^{k+1-i}, x_{n+1-i} = 1 - x_i (i = 1, \ldots, k)$, $x_{k+1} = 0.5$; here $\sigma \in (0, 0.5)$ is a constant. Of course, k is dependent on ε by the relation similar to (2.28). Error estimation can be obtained as the follows. First we use the same technique as that in Section 2.1 and get the decomposition of the exact solution, now the process is more complicated because we must define the boundary correctors near $x = 0$ and 1. Then we follow the routine in Section 2.2 and get results similar to Theorem 4, and the counterpart to (2.31) is

$$\|u - u_p\|_{1,\varepsilon} \leqslant C_N(\varepsilon^{1/4}\delta^p + p^{-N}).$$

3. The turning point case

$p(x)$ is supposed to have one simple zero on $[0, 1]$ which is located at $x = 0.5$. Hence, there is an internal layer. Let $w(x) = ((x - 0.5)^2 + \varepsilon)^\beta$ where $\beta \in (0, 0.5)$, which displays the typical internal layer behavior (cf. [5]). We will see that by proper distribution of meshpoints x_i in Δ, there holds results similar to Lemma 3. In order to achieve this, we will study the property of $w(x)$.

Through elementary calculation, for any non-negative integer k, we find

$$\frac{\mathrm{d}^k w}{\mathrm{d}x^k} = \sum_{i=0}^{[k/2]} 2^{k-2i}\beta\cdots(\beta-(k-i)+1)(x-0.5)^{k-2i}((x-0.5)^2+\varepsilon)^{\beta-(k-i)}\frac{k!}{i!(k-2i)!}. \tag{3.1}$$

Using (3.1) we get

$$\begin{aligned}
&\left|\frac{\mathrm{d}^k w}{\mathrm{d}x^k}\right| \leqslant C\left(\frac{5}{2}\right)k!((x-0.5)^2+\varepsilon)^{\beta-k/2},\\
&\left\{\int_0^1 \left|\frac{\mathrm{d}^k w}{\mathrm{d}x^k}\right|^2 (x-0.5)^{2(k-l)}\,\mathrm{d}x\right\}^{1/2} \leqslant C\left(\frac{5}{2}\right)^k k! f_l(\beta,\varepsilon),
\end{aligned} \tag{3.2}$$

where $f_l(\beta,\varepsilon)=\left\{\int_0^1[(x-0.5)^2+\varepsilon]^{2\beta-l}\,\mathrm{d}x\right\}^{1/2}$. (3.2) is similar to (2.19) which served as the basis for Lemma 3.

In Δ, let $n=2k+1, x_{k+1}=0.5$, $x_i=\frac{1}{2}(1-\sigma^i), x_{n+1-i}=\frac{1}{2}(1+\sigma^i)(i=1,\ldots,k)$, follow the routine in Lemma 3.

Lemma 5. *For $w(x)$, there exists $\phi(x)\in S(p,\sigma,n)$, such that $\phi(0)=w(0), \phi(1)=u(1)$, and*

$$\begin{aligned}
&\left\|\frac{\mathrm{d}^m(w-\phi)}{\mathrm{d}x^m}\right\|_0 \leqslant C\delta^p f_m(\beta,\varepsilon), \quad m=0,1,\\
&\left\|(x-0.5)\frac{\mathrm{d}(w-\phi)}{\mathrm{d}x}\right\|_0 \leqslant C\delta^p f_0(\beta,\varepsilon),
\end{aligned} \tag{3.3}$$

where $\delta\in(0,1)$ is a constant. Also there is an imposed condition on n.

From (1.5) and (2.1) we get $\forall v\in S_p$ $\|u-u_p\|_{1,\varepsilon}\leqslant C(\|u-v\|_{1,\varepsilon}+\|(x-0.5)(u-v)'\|_0)$. If the solution satisfies (3.2), then by Lemma 5, we get

$$\|u-u_p\|_{1,\varepsilon}\leqslant C\delta^p f_0(\beta,\varepsilon),$$

which shows the exponential rate of convergence. If we can get the decomposition of u in the form of (2.12), then we get a result similar to Theorem 4.

4. Numerical experiments

First we discuss the choice of mesh parameter σ in (1.3). It is not very rigorous, but it can give some guidance in practice.

The dimension of the finite element space S_p is $N_0=(n+1)p-1$. Here we pick the smallest n which satisfies (2.28). For any given constant $\rho\in(0,1)$, from $(\sqrt{F_{\min}})^p=\rho$, we have $p=\log\rho/\log\sqrt{F_{\min}}$. So approximately,

$$N_0 \approx \frac{2\log\rho(\log\varepsilon-\log\frac{d}{2})}{\log\sigma\log F_{\min}}+\log\rho\left(\frac{1}{\log\sigma}+\frac{2}{F_{\min}}\right)-1. \tag{4.1}$$

Table 1

σ	$G_2(\sigma)$	$G_1(\sigma)$
0.0010	$0.172521 \cdot 10^{4}$	$0.276587 \cdot 10^{-1}$
0.0100	$0.113986 \cdot 10^{3}$	$0.186055 \cdot 10^{0}$
0.1000	$0.522287 \cdot 10^{1}$	$0.101513 \cdot 10^{1}$
0.2000	$0.167228 \cdot 10^{1}$	$0.154896 \cdot 10^{1}$
0.3000	$0.775086 \cdot 10^{0}$	$0.187018 \cdot 10^{1}$
0.4000	$0.417022 \cdot 10^{0}$	$0.201330 \cdot 10^{1}$
0.5000	$0.240070 \cdot 10^{0}$	$0.200130 \cdot 10^{1}$
0.6000	$0.140457 \cdot 10^{0}$	$0.185782 \cdot 10^{1}$
0.7000	$0.794419 \cdot 10^{-1}$	$0.160138 \cdot 10^{1}$
0.8000	$0.401847 \cdot 10^{-1}$	$0.123910 \cdot 10^{1}$
0.9000	$0.146975 \cdot 10^{-1}$	$0.755285 \cdot 10^{0}$
0.9900	$0.840130 \cdot 10^{-3}$	$0.120231 \cdot 10^{0}$
0.9990	$0.603216 \cdot 10^{-4}$	$0.165944 \cdot 10^{-1}$

For fixed ρ, ε and d, N_0 is dependent on σ. So we use $N_0(\sigma)$ instead of N_0 to emphasize this relation. Now we look for $\sigma \in (0,1)$ such that it is the solution of the problem

$$\inf_{\sigma\in(0,1)} N_0(\sigma),$$

which means that we use the smallest number of unknowns to meet the given tolerance ρ.

Define

$$G_1(\sigma) = \log\sigma \log F_{\min}, \quad G_2(\sigma) = \frac{\log\sigma}{\log F_{\min}}.$$

Obviously, $G_2(\sigma)$ is a strictly decreasing function; moreover $\lim_{\sigma\to 0+} G_2(\sigma)=+\infty$, $\lim_{\sigma\to 1-} G_2(\sigma)=0$, $\lim_{\sigma\to 0+} G_1(\sigma)=0$, $\lim_{\sigma\to 1-} G_1(\sigma)=0$. From (4.1), we have $\lim_{\sigma\to 0+} N_0(\sigma)=+\infty$, $\lim_{\sigma\to 1-} N_0(\sigma)=+\infty$, so σ can not be in the small neighborhood of 0 and 1.

It is obvious that

$$\frac{1}{\log\sigma} + \frac{2}{\log F_{\min}} \geqslant \frac{2\sqrt{2}}{\sqrt{G_1(\sigma)}}$$

and the equality holds if and only if $G_2(\sigma) = \frac{1}{2}$. So if σ^* is the maximum point of $G_1(\sigma)$ and $G_2(\sigma^*) = \frac{1}{2}$, then it may be the minimum point of $N_0(\sigma)$. When $d = \alpha$, we give $G_1(\sigma)$ and $G_2(\sigma)$ for different σ in Table 1, from where we choose $\sigma = 0.4$.

We investigated the performance of the proposed finite element method by several numerical experiments. They were conducted on the problem

$$\begin{aligned} &-\varepsilon u'' - p(x)u' + p(x)u = f(x), \quad x \in (0,1), \\ &u(0) = u(1) = 0. \end{aligned} \tag{4.2}$$

In the following we let $\sigma = 0.4$.

Table 2

p	$\|e_p\|_0$	$\|e'_p\|_0$	$\|e_p\|_{1,\varepsilon}$	$e_p(1.e-8)$	$e_p(5.e-6)$
3	$0.809\cdot10^{-4}(-2.97)$	$0.393\cdot10^{1}(-1.80)$	$0.393\cdot10^{-2}(-1.80)$	$0.373\cdot10^{-04}(-2.29)$	$0.418\cdot10^{-03}(-1.88)$
4	$0.181\cdot10^{-5}(-3.38)$	$0.792\cdot10^{0}(-1.70)$	$0.792\cdot10^{-3}(-1.70)$	$0.271\cdot10^{-05}(-2.46)$	$0.164\cdot10^{-03}(-1.41)$
5	$0.115\cdot10^{-6}(-3.17)$	$0.155\cdot10^{0}(-1.68)$	$0.155\cdot10^{-3}(-1.68)$	$0.153\cdot10^{-06}(-2.59)$	$0.111\cdot10^{-04}(-1.84)$
6	$0.104\cdot10^{-7}(-2.98)$	$0.297\cdot10^{-1}(-1.67)$	$0.297\cdot10^{-4}(-1.67)$	$0.700\cdot10^{-08}(-2.72)$	$0.394\cdot10^{-05}(-1.64)$
7	$0.262\cdot10^{-8}(-2.66)$	$0.655\cdot10^{-2}(-1.64)$	$0.656\cdot10^{-5}(-1.64)$	$0.269\cdot10^{-09}(-2.83)$	$0.895\cdot10^{-07}(-2.07)$
8	$0.656\cdot10^{-9}(-2.45)$	$0.167\cdot10^{-2}(-1.59)$	$0.167\cdot10^{-5}(-1.59)$	$0.883\cdot10^{-11}(-2.92)$	$0.553\cdot10^{-07}(-1.80)$
9	$0.153\cdot10^{-9}(-2.31)$	$0.424\cdot10^{-3}(-1.56)$	$0.458\cdot10^{-6}(-1.55)$	$0.252\cdot10^{-12}(-3.01)$	$0.370\cdot10^{-09}(-2.26)$
10	$0.331\cdot10^{-10}(-2.21)$	$0.100\cdot10^{-3}(-1.55)$	$0.100\cdot10^{-6}(-1.55)$	$0.626\cdot10^{-14}(-3.10)$	$0.487\cdot10^{-09}(-1.94)$

4.1. The case of $p(x) > 0$

In this part, $p(x) = q(x) = 1$ and $n = 15$, ε is taken to be 10^{-6} unless specified otherwise.

Experiment 1. We take $f(x) = 1$. The exact solution of (4.2) is

$$u(x) = u_1(x) + u_2(x), \tag{4.3}$$

where

$$u_1(x) = \frac{e^{\lambda_2} - 1}{e^{\lambda_1} - e^{\lambda_2}} e^{\lambda_1 x}, \quad u_2(x) = \frac{1 - e^{\lambda_1}}{e^{\lambda_1} - e^{\lambda_2}} e^{\lambda_2 x} + 1 \text{ with } \lambda_1 = -\frac{1 + \sqrt{1 + 4\varepsilon}}{2\varepsilon}, \quad \lambda_2 = \frac{2}{1 + \sqrt{1 + 4\varepsilon}}.$$

From Lemma 2, we know that for $u_2(x)$ there exists $\phi_2(x) \in S(p, \sigma, n)$ such that $\phi_2(0) = u_2(0)$, $\phi_2(1) = u_2(1)$ and further we have

$$\|u_2 - \phi_2\|_1 \leqslant C \left(\frac{1}{(2p)!} \right)^{1/2} \left(\frac{\lambda_2}{2} \right)^p. \tag{4.4}$$

Define $e_p = u - u_p$, because $u_1(x) \in S$ with $d = \alpha = \frac{1}{2}(1 + \sqrt{1 + 4\varepsilon})$, using Lemma 3 and (4.4) we get

$$\|e_p\|_{1,\varepsilon} \leqslant C \left(\left(\sqrt{F_{\min}} \right)^p + \left(\frac{1}{(2p)!} \right)^{1/2} \left(\frac{\lambda_2}{2} \right)^p \right), \tag{4.5}$$

which shows exponential rate of convergence, the second part in the bound of (4.5) is trival compared with the first as $p \to +\infty$. To understand it more precisely, we define

$$r_p = \frac{\log \|e_p\|_{1,\varepsilon} - \log \|e_2\|_{1,\varepsilon}}{p - 2}. \tag{4.6}$$

The definition can be given similarly for $\|e'_p\|_0, \|e_p\|_0$ and $e_p(x)$. Table 2 gives some numerical results; the values in parentheses are the corresponding r_p's defined by (4.6).

In point of approximation, (2.20) implies that H^1-error is inversely proportional to $\sqrt{\varepsilon}$ while L^2-error is directly proportional. So we may pose the following problem: can we see this in e_p, the error of the finite element solution? For fixed S_p with $p = 8$, we let ε vary from 10^{-2} to 10^{-6} and report the results in Table 3, which shows the consistency of approximation. But we can not give

Table 3

ε	$\|e_p\|_{1,\varepsilon}$	$\sqrt{\varepsilon}\|e_p'\|_0$	$\|e_p\|_0/\sqrt{\varepsilon}$
$1.0 \cdot 10^{-1}$	$1.67{\cdot}10^{-6}$	$1.66{\cdot}10^{-6}$	$4.10{\cdot}10^{-7}$
$1.0 \cdot 10^{-2}$	$1.76{\cdot}10^{-6}$	$1.76{\cdot}10^{-6}$	$6.66{\cdot}10^{-7}$
$5.0 \cdot 10^{-3}$	$1.96{\cdot}10^{-6}$	$1.96{\cdot}10^{-6}$	$6.13{\cdot}10^{-7}$
$1.0 \cdot 10^{-3}$	$1.54{\cdot}10^{-6}$	$1.54{\cdot}10^{-6}$	$4.20{\cdot}10^{-7}$
$5.0 \cdot 10^{-4}$	$1.26{\cdot}10^{-6}$	$1.26{\cdot}10^{-6}$	$4.79{\cdot}10^{-7}$
$1.0 \cdot 10^{-4}$	$1.73{\cdot}10^{-6}$	$1.73{\cdot}10^{-6}$	$6.68{\cdot}10^{-7}$
$5.0 \cdot 10^{-5}$	$1.97{\cdot}10^{-6}$	$1.97{\cdot}10^{-6}$	$6.31{\cdot}10^{-7}$
$1.0 \cdot 10^{-5}$	$1.60{\cdot}10^{-6}$	$1.60{\cdot}10^{-6}$	$4.35{\cdot}10^{-7}$
$5.0 \cdot 10^{-6}$	$1.25{\cdot}10^{-6}$	$1.25{\cdot}10^{-6}$	$4.58{\cdot}10^{-7}$
$1.0 \cdot 10^{-6}$	$1.67{\cdot}10^{-6}$	$1.67{\cdot}10^{-6}$	$6.56{\cdot}10^{-7}$

Table 4

p	$\|e_p\|_0$	$\|e_p'\|_0$	$\|e_p\|_{1,\varepsilon}$	$e_p(1.e-8)$	$e_p(5.e-6)$
3	$0.722{\cdot}10^{-04}(-3.18)$	$0.393{\cdot}10^{01}(-1.80)$	$0.393{\cdot}10^{-02}(-1.80)$	$0.373{\cdot}10^{-04}(-2.29)$	$0.423{\cdot}10^{-03}(-1.84)$
4	$0.391{\cdot}10^{-05}(-3.05)$	$0.792{\cdot}10^{00}(-1.70)$	$0.792{\cdot}10^{-03}(-1.70)$	$0.272{\cdot}10^{-05}(-2.45)$	$0.165{\cdot}10^{-03}(-1.39)$
5	$0.804{\cdot}10^{-07}(-3.33)$	$0.155{\cdot}10^{00}(-1.68)$	$0.155{\cdot}10^{-03}(-1.68)$	$0.152{\cdot}10^{-06}(-2.60)$	$0.110{\cdot}10^{-04}(-1.83)$
6	$0.564{\cdot}10^{-07}(-2.58)$	$0.297{\cdot}10^{-01}(-1.67)$	$0.297{\cdot}10^{-04}(-1.67)$	$0.721{\cdot}10^{-08}(-2.71)$	$0.391{\cdot}10^{-05}(-1.63)$
7	$0.528{\cdot}10^{-08}(-2.54)$	$0.655{\cdot}10^{-02}(-1.64)$	$0.655{\cdot}10^{-05}(-1.64)$	$0.174{\cdot}10^{-09}(-2.91)$	$0.920{\cdot}10^{-07}(-2.05)$
8	$0.185{\cdot}10^{-08}(-2.29)$	$0.167{\cdot}10^{-02}(-1.59)$	$0.167{\cdot}10^{-05}(-1.59)$	$0.352{\cdot}10^{-10}(-2.69)$	$0.563{\cdot}10^{-07}(-1.79)$
9	$0.220{\cdot}10^{-09}(-2.27)$	$0.424{\cdot}10^{-03}(-1.56)$	$0.424{\cdot}10^{-06}(-1.56)$	$0.404{\cdot}10^{-10}(-2.29)$	$0.456{\cdot}10^{-09}(-2.23)$
10	$0.620{\cdot}10^{-10}(-2.14)$	$0.100{\cdot}10^{-03}(-1.55)$	$0.100{\cdot}10^{-06}(-1.55)$	$0.285{\cdot}10^{-10}(-2.05)$	$0.459{\cdot}10^{-09}(-1.95)$

the L^2-estimation for e_p except using $\|e_p\|_0 \leqslant \|e_p\|_{1,\varepsilon}$. Table 3 also verifies (4.5) by showing that approximately, $\|e_p\|_{1,\varepsilon}$ is independent of ε.

Experiment 2. The right-hand side $f(x)$ is chosen so that the exact solution of (4.2) is

$$u(x) = u_1(x) + u_2(x) + x^{\alpha}(1-x),$$

where $\alpha = 1.01$. Then $u(x)$ displays the singularity behavior in the neighborhood of $x = 0$. Numerical results are shown in Table 4. We are not surprised at the performance, because the idea of constructing the space S_p is similar to that of h–p version of the finite element which is suitable for handling singularity; see [10, 11]. But the existence of the singularity has some influence on the accuracy of u_p, this can be seen by comparison of Table 4 with Table 2.

Experiment 3. In the above two cases, the solutions are infinitely differentiable in $(0,1)$. Now we study further the performance of our method if $u(x)$ is not so smooth. This time $f(x)$ is chosen so that the solution of (4.2) is

$$u(x) = u_1(x) + u_2(x) + u_3(x), \tag{4.7}$$

Table 5

p	$\|e_p\|_0$	$\|e'_p\|_0$	$\|e_p\|_{1,\varepsilon}$	$e_p(1.e-8)$	$e_p(5.e-6)$
3	$0.363\cdot10^{-02}(-8.18)$	$0.414\cdot10^{01}(-5.82)$	$0.551\cdot10^{-02}(-7.37)$	$0.555\cdot10^{-04}(-6.75)$	$0.141\cdot10^{-02}(-8.78)$
4	$0.230\cdot10^{-02}(-5.44)$	$0.117\cdot10^{01}(-5.23)$	$0.258\cdot10^{-02}(-5.41)$	$0.937\cdot10^{-05}(-6.52)$	$0.104\cdot10^{-02}(-5.57)$
5	$0.124\cdot10^{-02}(-4.79)$	$0.493\cdot10^{00}(-4.90)$	$0.133\cdot10^{-02}(-4.81)$	$0.674\cdot10^{-05}(-5.29)$	$0.669\cdot10^{-03}(-4.70)$
6	$0.473\cdot10^{-03}(-4.88)$	$0.183\cdot10^{00}(-4.99)$	$0.507\cdot10^{-03}(-4.89)$	$0.253\cdot10^{-05}(-5.30)$	$0.257\cdot10^{-03}(-4.79)$
7	$0.491\cdot10^{-04}(-6.08)$	$0.124\cdot10^{-01}(-6.52)$	$0.512\cdot10^{-04}(-6.12)$	$0.145\cdot10^{-06}(-6.93)$	$0.143\cdot10^{-04}(-6.50)$
8	$0.154\cdot10^{-03}(-4.67)$	$0.576\cdot10^{-01}(-4.79)$	$0.164\cdot10^{-03}(-4.69)$	$0.809\cdot10^{-06}(-5.03)$	$0.808\cdot10^{-04}(-4.63)$
9	$0.152\cdot10^{-03}(-4.32)$	$0.580\cdot10^{-01}(-4.41)$	$0.162\cdot10^{-03}(-4.33)$	$0.815\cdot10^{-06}(-4.63)$	$0.813\cdot10^{-04}(-4.26)$
10	$0.756\cdot10^{-04}(-4.47)$	$0.290\cdot10^{-01}(-4.55)$	$0.813\cdot10^{-04}(-4.48)$	$0.407\cdot10^{-06}(-4.75)$	$0.407\cdot10^{-04}(-4.41)$

where

$$u_3(x) = \begin{cases} 1 + \left(\dfrac{x}{\beta} - 1\right)^3, & x \in (0,\beta), \\ 1, & x \in \left(\beta, \dfrac{1+\beta}{2}\right), \\ 1 + \left(\dfrac{1+\beta}{1-\beta} - \dfrac{2}{1-\beta}x\right)^3, & x \in \left(\dfrac{1+\beta}{2}, 1\right), \end{cases} \tag{4.8}$$

with $\beta = 0.01$. Now β and $\frac{1}{2}(1+\beta)$ are not mesh points.

We can introduce a linear operator $P_\varepsilon : H^1(0,1) \to S(p,\sigma,n)$, defined by

$$B(v - P_\varepsilon v, v_p) = 0, \quad \forall v \in H^1(0,1), \forall v_p \in S_p;$$

moreover we demand that $P_\varepsilon v(0) = v(0)$ and $P_\varepsilon v(1) = v(1)$. Then we obtain the following decomposition for e_p:

$$e_p = u_1 + u_2 - P_\varepsilon(u_1 + u_2) + u_3 - P_\varepsilon u_3 \equiv e_p^1 + e_p^2, \tag{4.9}$$

from which we know that e_p^2 will dominate for large p and only the polynomial rate of convergence will be obtained. To see it more precisely, we define

$$r_p = \frac{\log \|e_p\|_{1,\varepsilon} - \log \|e_2\|_{1,\varepsilon}}{\log p - \log 2}. \tag{4.10}$$

Similarly, this definition can be extended to the cases of $\|e'_p\|_0$, $\|e_p\|_0$ and $e_p(x)$. Computed results are displayed in Table 5; the values in parentheses are the corresponding r_p defined by (4.10). Because $u_3(x) \in H^{3.5}(0,1)$, from [1], the rate of convergence is 2.5 for $\|e_p^2\|_1$ and 3.5 for $\|e_p^2\|_0$; this has been obtained in our practical computation when we let $\varepsilon = 1$. But now in Table 5, the rates are surprisingly higher. Table 6 shows some results corresponding to e_p^2, and comparing it with Table 5 verifies the fact that e_p^2 dominates the error e_p.

In order to show the relation between $\|e_p\|_{1,\varepsilon}$ and ε, we fix the space S_p with $p = 9$, and let ε vary from 0.1 to 10^{-6}. Computed results are given in Table 7, from which we obtain that for small ε, $\|e_p\|_{1,\varepsilon}$ is approximately independent of ε.

Table 6

p	$\|e_p^2\|_0$	$\|(e_p^2)'\|_0$	$\|e_p^2\|_{1,\varepsilon}$
3	$0.355\cdot10^{-2}$	$0.130\cdot10^{1}$	$0.378\cdot10^{-2}$
4	$0.230\cdot10^{-2}$	$0.859\cdot10^{-0}$	$0.246\cdot10^{-2}$
5	$0.124\cdot10^{-2}$	$0.468\cdot10^{-0}$	$0.132\cdot10^{-2}$
6	$0.473\cdot10^{-3}$	$0.180\cdot10^{-0}$	$0.506\cdot10^{-3}$
7	$0.491\cdot10^{-4}$	$0.105\cdot10^{-1}$	$0.502\cdot10^{-4}$
8	$0.154\cdot10^{-3}$	$0.576\cdot10^{-1}$	$0.164\cdot10^{-3}$
9	$0.152\cdot10^{-3}$	$0.580\cdot10^{-1}$	$0.162\cdot10^{-3}$
10	$0.756\cdot10^{-4}$	$0.290\cdot10^{-1}$	$0.809\cdot10^{-4}$

Table 7

ε	$\|e_p\|_{1,\varepsilon}$	$\|e_p\|_0$	$\|e_p'\|_0$
$1.0\cdot10^{-1}$	$5.23\cdot10^{-4}$	$3.51\cdot10^{-5}$	$1.65\cdot10^{-3}$
$1.0\cdot10^{-2}$	$2.14\cdot10^{-4}$	$5.07\cdot10^{-5}$	$2.08\cdot10^{-3}$
$5.0\cdot10^{-3}$	$1.88\cdot10^{-4}$	$7.87\cdot10^{-5}$	$2.41\cdot10^{-3}$
$1.0\cdot10^{-3}$	$1.67\cdot10^{-4}$	$1.29\cdot10^{-4}$	$3.24\cdot10^{-3}$
$5.0\cdot10^{-4}$	$1.63\cdot10^{-4}$	$1.38\cdot10^{-4}$	$3.75\cdot10^{-3}$
$1.0\cdot10^{-4}$	$1.60\cdot10^{-4}$	$1.46\cdot10^{-4}$	$6.35\cdot10^{-3}$
$5.0\cdot10^{-5}$	$1.60\cdot10^{-4}$	$1.48\cdot10^{-4}$	$8.55\cdot10^{-3}$
$1.0\cdot10^{-5}$	$1.62\cdot10^{-4}$	$1.51\cdot10^{-4}$	$1.85\cdot10^{-2}$
$5.0\cdot10^{-6}$	$1.62\cdot10^{-4}$	$1.51\cdot10^{-4}$	$2.60\cdot10^{-2}$
$1.0\cdot10^{-6}$	$1.62\cdot10^{-4}$	$1.52\cdot10^{-4}$	$5.80\cdot10^{-2}$

Table 8

p	$\|e_p\|_0$	$\|e_p'\|_0$	$\|e_p\|_{1,\varepsilon}$
3	$0.264\cdot10^{-04}(-1.73)$	$0.850\cdot10^{00}(-1.79)$	$0.890\cdot10^{-04}(-1.78)$
4	$0.518\cdot10^{-05}(-1.68)$	$0.172\cdot10^{00}(-1.69)$	$0.179\cdot10^{-04}(-1.69)$
5	$0.969\cdot10^{-06}(-1.68)$	$0.335\cdot10^{-01}(-1.67)$	$0.349\cdot10^{-05}(-1.67)$
6	$0.215\cdot10^{-06}(-1.64)$	$0.645\cdot10^{-02}(-1.67)$	$0.680\cdot10^{-06}(-1.66)$
7	$0.549\cdot10^{-07}(-1.58)$	$0.146\cdot10^{-02}(-1.63)$	$0.156\cdot10^{-06}(-1.63)$
8	$0.138\cdot10^{-07}(-1.55)$	$0.374\cdot10^{-03}(-1.59)$	$0.398\cdot10^{-07}(-1.58)$
9	$0.320\cdot10^{-08}(-1.54)$	$0.938\cdot10^{-04}(-1.56)$	$0.992\cdot10^{-08}(-1.55)$
10	$0.684\cdot10^{-09}(-1.54)$	$0.218\cdot10^{-04}(-1.54)$	$0.229\cdot10^{-08}(-1.54)$

4.2. The case of $p(x) \equiv 0$

In this part, $q(x) = 1, n = 21, \varepsilon = 10^{-8}$. The exact solution of (4.2) is

$$u(x) = \frac{1 - e^{1/\sqrt{\varepsilon}}}{e^{1/\sqrt{\varepsilon}} - e^{-1/\sqrt{\varepsilon}}}\left(e^{(x-1)/\sqrt{\varepsilon}} + e^{-x/\sqrt{\varepsilon}}\right) + 1.$$

Numerical results are shown in Table 8; the definition of the values in parentheses is similar to that given by (4.6). It is also the same in the following.

Table 9

$\beta = 0.01$

p	$\Vert e_p \Vert_0$	$\Vert e_p' \Vert_0$	$\Vert e_p \Vert_{1,\varepsilon}$
3	$0.679 \cdot 10^{-04}(-3.50)$	$0.139 \cdot 10^{01}(-1.78)$	$0.155 \cdot 10^{-03}(-2.74)$
4	$0.952 \cdot 10^{-04}(-1.58)$	$0.430 \cdot 10^{00}(-1.48)$	$0.104 \cdot 10^{-03}(-1.57)$
5	$0.453 \cdot 10^{-05}(-2.07)$	$0.113 \cdot 10^{00}(-1.43)$	$0.122 \cdot 10^{-04}(-1.76)$
6	$0.479 \cdot 10^{-05}(-1.54)$	$0.266 \cdot 10^{-01}(-1.44)$	$0.548 \cdot 10^{-05}(-1.52)$
7	$0.109 \cdot 10^{-06}(-1.99)$	$0.511 \cdot 10^{-02}(-1.48)$	$0.523 \cdot 10^{-06}(-1.69)$
8	$0.238 \cdot 10^{-06}(-1.53)$	$0.121 \cdot 10^{-02}(-1.47)$	$0.268 \cdot 10^{-06}(-1.52)$
9	$0.169 \cdot 10^{-07}(-1.69)$	$0.181 \cdot 10^{-03}(-1.53)$	$0.247 \cdot 10^{-07}(-1.64)$
10	$0.112 \cdot 10^{-07}(-1.53)$	$0.553 \cdot 10^{-04}(-1.49)$	$0.125 \cdot 10^{-07}(-1.52)$

Table 10

$\beta = 0.5$

p	$\Vert e_p \Vert_0$	$\Vert e_p' \Vert_0$	$\Vert e_p \Vert_{1,\varepsilon}$
3	$0.274 \cdot 10^{-03}(-2.59)$	$0.176 \cdot 10^{-01}(-1.89)$	$0.274 \cdot 10^{-03}(-2.59)$
4	$0.977 \cdot 10^{-04}(-1.81)$	$0.559 \cdot 10^{-02}(-1.52)$	$0.977 \cdot 10^{-04}(-1.81)$
5	$0.852 \cdot 10^{-05}(-2.02)$	$0.835 \cdot 10^{-03}(-1.64)$	$0.852 \cdot 10^{-05}(-2.02)$
6	$0.348 \cdot 10^{-05}(-1.74)$	$0.278 \cdot 10^{-03}(-1.51)$	$0.348 \cdot 10^{-05}(-1.74)$
7	$0.314 \cdot 10^{-06}(-1.87)$	$0.385 \cdot 10^{-04}(-1.60)$	$0.314 \cdot 10^{-06}(-1.87)$
8	$0.136 \cdot 10^{-06}(-1.70)$	$0.136 \cdot 10^{-04}(-1.51)$	$0.136 \cdot 10^{-06}(-1.70)$
9	$0.125 \cdot 10^{-07}(-1.80)$	$0.196 \cdot 10^{-05}(-1.57)$	$0.125 \cdot 10^{-07}(-1.80)$
10	$0.560 \cdot 10^{-08}(-1.67)$	$0.702 \cdot 10^{-06}(-1.50)$	$0.560 \cdot 10^{-08}(-1.67)$

4.3. The case of turning point

In this part, $p(x) = (x-0.5)/\beta + 0.3121(x-0.5)^2/\beta$, $q(x) = 1 + 0.2764(x-0.5)$. This example is from [5]. We let $n = 21$ and $\varepsilon = 10^{-8}$. Define

$$y(x) = (0.291(x-0.5)^2 + \varepsilon)^{\beta/2} + (0.291(x-0.5)^2 + \varepsilon)^{(\beta-1)/2}(x-0.5) + e^{-0.5x^2},$$

$f(x)$ is so taken that $u(x) = y(x) - y(1)x - y(0)(1-x)$. So $u(x)$ satisfies (3.2), and then the exponential rate of convergence can be achieved, see Tables 9 and 10 for details. From [8], we know that several kinds of difference schemes are uniformly convergent of order $h^{\min\{\beta,1\}}$. Now the convergence rate is much higher.

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